October Math Gems

Problem of the week 16

§1 Problems

Problem 1.1. Let $\mathbb{Z}$ be the set of integers. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,
$$f(2a) + 2f(b) = f(f(a + b)).$$

answer. By taking $a = 0$, we get $f(0) + 2f(b) = f(f(b))$ for all $b \in \mathbb{Z}$. So, the problem becomes:
$$f(2a) + 2f(b) = f(0) + 2f(a + b)$$

Taking $a = 1$ for this one, we quickly obtain that $2(f(b + 1) - f(b)) = f(2) - f(0)$, so f is linear. The rest is checking together with $f(0) + 2f(b) = f(f(b))$. $\square$

Problem 1.2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$xy(f(x + y) - f(x) - f(y)) = 2f(xy)$$
for all $x, y \in \mathbb{R}$.

answer. Let $P(x, y)$ be the assertion $xy(f(x + y) - f(x) - f(y)) = 2f(xy)$ Let $c = f(1)$
$P(0, 0) \implies f(0) = 0$ $P(-2, 1) \implies f(-1) = c$
$P(x - 1, 1) \implies f(x - 1) = \frac{x-1}{x+1}f(x) - c\frac{x-1}{x+1}$ $\forall x \neq -1$
$P(x, -1) \implies f(x - 1) = f(x) + c - \frac{2}{x}f(-x)$ $\forall x \neq 0$
And so (subtracting) $\frac{x-1}{x+1}f(x) - c\frac{x-1}{x+1} = f(x) + c - \frac{2}{x}f(-x)$ $\forall x \neq -1, 0$ Which is $xf(x) - (x + 1)f(-x) = -cx^2$ $\forall x \neq -1, 0$, still true when $x = 0$ or $x = -1$
Moving $x \to -x$, we get $-xf(-x) + (x - 1)f(x) = -cx^2$ Cancelling $f(-x)$ between the two lines, we get : $\boxed{f(x) = cx^2 \quad \forall x}$, which indeed fits, whatever is $c \in \mathbb{R}$ $\square$

Problem 1.3. In a triangle ABC with $sinA = cosB = cotC$. Find the value of $cosA$.

answer.
$$A = \frac{\pi}{2} + B$$
$$sinA = \cot C = \cot\left(\frac{3\pi}{2} - 2A\right) = \tan 2A$$
$$2\cos A = \cos 2A = 2\cos^2 A - 1$$
$$\cos A = \frac{1 - \sqrt{3}}{2}$$
$\square$

Problem 1.4.

$$cos^{-1}\frac{\sqrt{27+7\sqrt{5}}+\sqrt{27-7\sqrt{5}}}{14}$$

answer. Let

$$S=\frac{\sqrt{27+7\sqrt{5}}+\sqrt{27-7\sqrt{5}}}{14}$$

$$S^2=\frac{1}{196}(27+7\sqrt{5}+27-7\sqrt{5}+2\cdot 22)=\frac{1}{2}$$

Thus, $S=\frac{\sqrt{2}}{2}$ which $\cos^{-1}\frac{\sqrt{2}}{2}=\frac{\pi}{4}$. $\square$

Problem 1.5. Find all triples (x, y, z) of real numbers that satisfy the system of equations

$$\begin{cases} x^3 = 3x - 12y + 50, \\ y^3 = 12y + 3z - 2, \\ z^3 = 27z + 27x. \end{cases}$$

answer. Suppose that $x > 2$. Then on $(2, \infty)$ the function $x^3 - 3x$ is increasing so we have $y < 4$ by the first equation.
Also the third equation yields that, from $x > 2$, $z^3 - 27z > 54 \implies z > 6$. But then the second equation tells us that $y^3 - 12y > 16 \implies y > 4$. Contradiction.
Now suppose that $x < 2$. From equation 3, $z^3 - 27z < 54 \implies z < 6$. Then equation 2 yields $y < 4$. Using this in the first equation we have $x^3 - 3x = 50 - 12y > 2 \implies x > 2$. Contradiction again.
It is easy to check that $x = 2, y = 4, z = 6$ works so $(x, y, z) = \boxed{2, 4, 6}$ $\square$

Problem 1.6. Let a and b be real numbers such that $\sin a+\sin b=\dfrac{\sqrt{2}}{2}$ and $\cos a+\cos b=\dfrac{\sqrt{6}}{2}$. Find $\sin(a+b)$.

answer. $\sin a+\sin b=2\sin\frac{a+b}{2}\cos\frac{a-b}{2},\ \cos a+\cos b=2\cos\frac{a+b}{2}\cos\frac{a-b}{2}$
$\frac{1}{\sqrt{3}}=\frac{\sin a+\sin b}{\cos a+\cos b}=\tan\frac{a+b}{2},\ \frac{a+b}{2}=\frac{\pi}{6}+n\pi\ (n\in\mathbb{Z})$
$\sin(a+b)=\sin\left(\frac{\pi}{3}+2n\pi\right)=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$ $\square$

Problem 1.7. In triangle $ABC, AB = \sqrt{30}, AC = \sqrt{6}$, and $BC = \sqrt{15}$. There is a point D for which $\overline{AD}$ bisects $\overline{BC}$ and $\angle ADB$ is a right angle. The ratio

$$\frac{\text{Area}(\triangle ADB)}{\text{Area}(\triangle ABC)}$$

can be written in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.

answer. By Stewart's Theorem, we can find AM, which is $\frac{\sqrt{57}}{2}$. Obviously, as stated in the problem, $CM = MB = \frac{\sqrt{15}}{2}$. Let $MD = x$. We can state that:
$(\frac{\sqrt{57}}{2}+x)^2+BD^2=30\ (x^2+BD^2)=\sqrt{154}$
Solving this dreadful equation, we end up having $x=\frac{4\sqrt{57}}{19}$. Since $[ACM] = [AMB]$ because these triangles have the bases of same length and they have the same height, all we need to do is to find the ratio:
$\frac{1}{2}+\frac{[BED]}{2[ABE]}$
Simplifying, and after a very happy and exciting result at the end when everything cancels out, we end up with 27/38. So, our final answer is $27 + 38 = \boxed{65}$. $\square$

Problem 1.8. find area of triangle with sides $\sqrt{a^2+b^2}, \sqrt{c^2+a^2}, \sqrt{b^2+c^2}$, given a,b,c are positive. PS:- (don't use heron's formula, and $\frac{1}{2}$product of sides and product of sine of angle between these sides

answer. Points $A(0,0), B(\sqrt{a^2+b^2}, 0)$,
then $C(\frac{a^2}{\sqrt{a^2+b^2}}, \frac{\sqrt{a^2b^2+b^2c^2+c^2a^2}}{\sqrt{a^2+b^2}})$.
Area $= \frac{1}{2}\sqrt{a^2b^2+b^2c^2+c^2a^2}$. □

Problem 1.9. Let ABC be a triangle such that $\angle B = 20^\circ$ and $\angle C = 40^\circ$. Also, length of bisection of angle A is equal to 2. Find the value of $BC - AB$?

answer. Clearly $\angle A = 120^\circ$. Let D be the intersection of the angle bisector through A with BC and E the point on BC with $\angle CAE = 40^\circ$.
Then $\angle EAB = 80^\circ = \angle BEA$ and thus $BE = AB$, i.e. $CE = BC - AB$.
On the other hand, $\angle CAE = \angle ECA$ implies that $CE = AE$.
Finally, $\angle ADC = 180^\circ - 60^\circ - 40^\circ = 80^\circ = \angle BEA$, which implies $AE = AD$. Summarising, we get $BC - AB = AD = 2$. □

Problem 1.10. Given acute $\triangle ABC$. Given point N such that$\angle NBA = \angle NCA = 90^\circ$. D and E are points on AC and AB, respectivelly, such that$\angle BNE = \angle CND$.Lines DE and BC intersects in F, and K is midpoint of segment DE. If X is point of intersection of circumcircles of $\triangle ABC$ and $\triangle ADE$, distinct from A, prove that $\angle KXF = 90^\circ$.

answer. X is the Miquel Point of the quadrilateral $BEDC$ (i.e. $X \equiv \odot(AED) \cap \odot(DCF) \cap \odot(ABC) \cap \odot(BEF)$)
Then X is the spiral center on $BE \to CD \Rightarrow \frac{EX}{DX} = \frac{BE}{CD}$
Since $\triangle NBE \sim \triangle NCD \Rightarrow \frac{BN}{NC} = \frac{BE}{CD} \Rightarrow \frac{BN}{NC} = \frac{XE}{XC}$
Let G be the symmetrical of X with respect to $K \Rightarrow \angle GDX = 180^\circ - \angle BAC = \angle BNC \Rightarrow \triangle GDX \sim \triangle BNC \Rightarrow \angle KXD = \angle BCN$ and $\angle DCB = \angle DXF \Rightarrow \angle KXF = 90^\circ$ as desired. □

Problem 1.11. Determine all roots, real or complex, of the system of simultaneous equations

$$x+y+z=3,$$
$$x^2+y^2+z^2=3,$$
$$x^3+y^3+z^3=3.$$

answer. We begin by manipulating the given equations to find other useful relations between the variables. $(\sum x)^2 = \sum x^2 + 2\sum xy$ $3^2 = 3 + 2\sum xy$ $\sum xy = 3$
Now using the following identity, $\sum x^3 - 3xyz = (\sum x)(\sum x^2 - \sum xy)$ $3 - 3xyz = 3(0)$ $xyz = 1$
Now, let x, y, and z be the roots of a cubic polynomial $P(t)$. Then, by Vieta's Formulas, we have that $P(t) = t^3 - 3t^2 + 3t - 1$ $P(t) = (t-1)^3$
Therefore, $\boxed{x = y = z = 1}$ is the only solution. □

Problem 1.12. Find $ax^5 + by^5$ if the real numbers a, b, x, and y satisfy the equations

$$\begin{aligned} ax + by &= 3, \\ ax^2 + by^2 &= 7, \\ ax^3 + by^3 &= 16, \\ ax^4 + by^4 &= 42. \end{aligned}$$

answer. Set $S = (x + y)$ and $P = xy$. Then the relationship

$$(ax^n + by^n)(x + y) = (ax^{n+1} + by^{n+1}) + (xy)(ax^{n-1} + by^{n-1})$$

can be exploited:

$$\begin{aligned} (ax^2 + by^2)(x + y) &= (ax^3 + by^3) + (xy)(ax + by) \\ (ax^3 + by^3)(x + y) &= (ax^4 + by^4) + (xy)(ax^2 + by^2) \end{aligned}$$

Therefore:

$$\begin{aligned} 7S &= 16 + 3P \\ 16S &= 42 + 7P \end{aligned}$$

Consequently, $S = -14$ and $P = -38$.
Finally:

$$\begin{aligned} (ax^4 + by^4)(x + y) &= (ax^5 + by^5) + (xy)(ax^3 + by^3) \\ (42)(S) &= (ax^5 + by^5) + (P)(16) \\ (42)(-14) &= (ax^5 + by^5) + (-38)(16) \\ ax^5 + by^5 &= 20 \end{aligned}$$

□

Problem 1.13. Find all real numbers a, b, c, d such that

$$\begin{cases} a + b + c + d = 20, \\ ab + ac + ad + bc + bd + cd = 150. \end{cases}$$

answer. Let $S = ab + ac + ad + bc + bd + cd = 150$. We know that $(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2S$, so $a^2 + b^2 + c^2 + d^2 = 400 - 300 = 100$.
So we have to find all real numbers a, b, c, d such that (1) $a + b + c + d = 20$, and (2) $a^2 + b^2 + c^2 + d^2 = 100$.
By AM-QM inequality we have that $\frac{a+b+c+d}{4} \leq \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}$, with equality if and only if $a = b = c = d$.
But these expressions are in fact equal, as they both equal 5. So the only solution is $a = b = c = d = 5$ □

Problem 1.14. Determine all real numbers $x > 0$ for which

$$\log_4 x - \log_x 16 = \frac{7}{6} - \log_x 8$$

answer. We have given the equation (1) $\log_4 x - \log_x 16 = \frac{7}{6} - \log_x 8$, Clearly $x > 0$ by equation (1), which implies there is a real number y s.t. (2) $x = 2^y$. The fact that $x = (2^n)^{\frac{y}{n}}$ when $n \in \mathbb{N}$ yields $\log_{2^n} x = \frac{y}{n}$, which combined with equation (1) give us $\frac{y}{2} - \log_x 2^4 = \frac{7}{6} - \frac{y}{3}$, i.e. (3) $5y - 7 = 24 \log_x 2$. Now there is a real number z s.t. $x^z = 2$, which according to formula (2) means $2^{yz} = 2$, implying $yz = 1$. The fact that $x^z = 2$ implies $\log_x 2 = z = \frac{1}{y}$, which inserted in equation (3) result in $5y - 7 = \frac{24}{y}$, yielding $5y^2 - 7y + 24 = 0$, i.e. $(5y + 8)(y - 3) = 0$. Consequently $y = -\frac{8}{5}$ or $y = 3$, which according to formula (2) means there are exactly two solutions of equation (1), namely $x = 2^{-\frac{8}{5}} = \frac{1}{\sqrt[5]{256}}$ and $x = 2^3 = 8$. □

Problem 1.15. Solve in the real numbers the equation $3^{\sqrt[3]{x-1}}\left(1 - \log_3^3 x\right) = 1$.

answer. Assuming $\log_3^3 x = (\log_3 x)^3$ we have :

$$3^{\left(\frac{3^t-1}{3^t}\right)^{\frac{1}{3}}} = t^3 + 1$$

where $t = \sqrt[3]{x-1}$ Let $k = \left(\frac{3^t-1}{3^t}\right)^{\frac{1}{3}}$

$$\implies (t^3 + 1)^t = (1 - k^3)^{-k}$$

Let $k = -b$

$$\implies (t^3 + 1)^t = (b^3 + 1)^b$$

Now if $b > t$ then $LHS < RHS$ and if $b < t$ then $LHS > RHS$ So , we must have $b = t$

$$\implies t = -k$$

$$\implies \sqrt[3]{x-1} = -\left(\frac{3^t - 1}{3^t}\right)^{\frac{1}{3}}$$

$$\implies (x) = \frac{1}{3^{\sqrt[3]{x-1}}}$$

Now note LHS is increasing and RHS is decreasing. So, atmost one solution. We can easily check that 1 satisfies. □

Problem 1.16. Evaluate the following sum

$$\frac{1}{\log_2 \frac{1}{7}} + \frac{1}{\log_3 \frac{1}{7}} + \frac{1}{\log_4 \frac{1}{7}} + \frac{1}{\log_5 \frac{1}{7}} + \frac{1}{\log_6 \frac{1}{7}} - \frac{1}{\log_7 \frac{1}{7}} - \frac{1}{\log_8 \frac{1}{7}} - \frac{1}{\log_9 \frac{1}{7}} - \frac{1}{\log_{10} \frac{1}{7}}$$

answer. Just use $\log_a b \cdot \log_b a = 1$. The sum is:

$$\log_{\frac{1}{7}} \frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{7 \cdot 8 \cdot 9 \cdot 10} = 1$$

□

Problem 1.17. Solve in $\mathbb{R}$ the equation: $\log_{278}(\sqrt{x} + \sqrt[4]{x} + \sqrt[8]{x} + \sqrt[16]{x}) = log_2 \sqrt[16]{x}$.

answer. Write $x = y^{16} > 0$ and $t = \log_2 278 > 8$ and equation is : $y + y^2 + y^4 + y^8 = y^t$ Or also $1 + \frac{1}{y^4} + \frac{1}{y^6} + \frac{1}{y^7} = y^{t-8}$ For $y > 0$, LHS is continuous strictly decreasing while RHS is continuous strictly increasing and so at most one positive real root. And since $y = 2$ is a trivial one, this is the only one. Hence the unique solution $\boxed{x = 2^{16}}$ □

Problem 1.18. Solve in real numbers: $log_{\sqrt{x}}(log_2(4^x - 2)) \leq 2$

answer. If $x > 1$: $4^x - 2^x - 2 = (2^x + 1)(2^x - 2) > 0 \implies 4^x - 2 > 2^x \implies \log_2(4^x - 2) > x \implies \log_2(\log_2(4^x - 2)) > 2\log_2\sqrt{x} > 0 \implies \log_{\sqrt{x}}(\log_2(4^x - 2)) > 2$
And so no solution If $x = 1$: $\log_{\sqrt{x}}(...)$ is undefined
And so no solution If $1 > x > \log_2\sqrt{3}$: $4^x - 2^x - 2 = (2^x + 1)(2^x - 2) < 0 \implies 1 < 4^x - 2 < 2^x \implies 0 < \log_2(4^x - 2) < x \implies \log_2(\log_2(4^x - 2)) > 2\log_2\sqrt{x} < 0 \implies \log_{\sqrt{x}}(\log_2(4^x - 2)) > 2$
And so no solution If $\log_2\sqrt{3} \geq x > \frac{1}{2}$ $4^x - 2^x - 2 = (2^x + 1)(2^x - 2) < 0 \implies 0 < 4^x - 2 \leq 1 \implies \log_2(4^x - 2) < 0 \implies \log_2(\log_2(4^x - 2))$ undefined
And so no solution If $\frac{1}{2} \geq x$ $4^x - 2 \leq 0 \implies \log_2(4^x - 2)$ undefined
And so no solution
Hence the result : $\boxed{\text{No solution}}$ □

Problem 1.19. Find all positive integers a, b such that

$$\frac{a^b + b^a}{a^a - b^b}$$

is an integer.

answer. Let $a > b \geq 1$ Then $a^b + b^a \geq a^a - b^b$ or $b^a + b^b \geq a^a - a^b$ $(b+1)^a > b^a + b^b$ and $a^a - a^b = \frac{2a^a - 2a^b}{2} \geq \frac{a^a}{2}$ So $(b+1)^a > \frac{a^a}{2}$ or $(1 + \frac{1}{b+1})^a \geq (1 + \frac{1}{b+1})^{b+1} > 2 > (\frac{a}{b+1})^a$ or $b + 2 > a$ to $a = b + 1$ and then $b^{b+1} + b^b \geq (b+1)^{b+1} - (b+1)^b = b * (b+1)^b \geq 2b^{b+1} \rightarrow b = 1, a = 2$ □

Problem 1.20. Find x if $\sqrt{x} = \sqrt[\sqrt{x}^{\sqrt{x}^5}]{5}$

answer. $x = 5^{2y} \implies y = 5^{-y5^{5y}}$ $y = 5^{-z} \implies z = 5^{5^{1-z}-z}$ $z = 5^u \implies u + 5^u = 5^{1-5^u}$
LHS is continuous increasing while RHS is continuous decreasing and so at most one solution. And since $u = 0$ is a trivial root, this is the unique one Back to x, we get $\boxed{x = 5^{\frac{2}{5}}}$ as unique solution. □
